Derivation of the formula for the cone angle ε for a constant chord blade which is connected to the hub by a hinge

ing. A. Kragten

October 2019
reviewed December 2024

KD 684

Engineering office Kragten Design
Populierenlaan 51
5492 SG Sint-Oedenrode
The Netherlands
telephone: +31 413 475770
e-mail: info@kdwindturbines.nl
website: www.kdwindturbines.nl

Contains page

1 Introduction

Report KD 35 (ref. 1) gives the procedure how to design the blades of a wind turbine. This report gives only the blade geometry and it isn’t checked in KD 35 if the designed blade is strong enough. However, I have written many KD-reports in which the design calculations are given for a certain rotor and in some of these reports the strength calculation is given too. All public KD-reports with design calculations of a specific rotor are mentioned in the note: “Sequence of KD-reports for self-study” which can be found at the top of the list with KD-reports on my website. The maximum bending stress normally occurs in the connecting strip which connects a blade to the hub, just at the edge of the hub. Stress calculations at this position are given in chapter 8 of report KD 484 (ref. 2) for the 3-bladed VIRYA-3B3 rotor.

The blade is pushed backwards because of the rotor thrust acting on each blade and this gives a bending moment in the connecting strip. If the connecting strip is rather thin, the blade connection is rather flexible and this makes that the blade bends backwards somewhat. If the blade has bent backwards, the centrifugal force in the blade gives a bending moment which bends the blade forwards. So this bending moment of the centrifugal force reduces the bending moment which is caused by the thrust. This also reduces the bending stress.

For the calculation of the final bending stress it is necessary to know how far the blade moves backwards if the blade is connected to the hub by a hinge. The cone angle in between the position of the blade and the plane perpendicular to the rotor axis is called ε. Already in 1980, when I was working in the Wind Energy Group of the University of Eindhoven, I have derived a formula for ε in report R 409 D (ref. 3). However, this report is in Dutch and no longer available and therefore it seems useful to give the derivation in this new public report KD 684. Formula 6 of R 409 D is valid for a blade with a constant chord and a constant airfoil but most blades of the VIRYA wind turbines which I have designed, have such blades.

2 Description of the rotor thrust

The aerodynamic force acting on a small blade section of a rotating blade of a wind turbine is build up by the lift force ΔL and the drag force ΔD. Both ΔL and ΔD are acting in the aerodynamic centre of the airfoil which is situated at 0.25 c from the nose of the airfoil. Apart from lift and drag there is also an aerodynamic moment acting on the airfoil but this moment isn’t taken into account because it is assumed that the rotor has fixed blades. Formula 4.14 of KD 35 (ref. 1) gives the thrust force ΔT perpendicular to the rotor plane for a small blade section. This formula is copied as formula 1.

$$\Delta T = \Delta L \cos \phi + \Delta D \sin \phi \qquad \text{(N)} \qquad (1)$$

In this formula ϕ is the angle in between the relative wind W and the rotor plane and ϕ is rather small for a rotor with a high design tip speed ratio λ_d. As ΔL is large with respect to ΔD and as $\cos \phi$ is large with respect to $\sin \phi$, the lift force ΔL gives the major contribution to ΔT. As ΔL and ΔD are acting in the aerodynamic centre of the airfoil this is also the case for ΔT. The rotor thrust force F_t is the sum of all thrust forces ΔT for all blade sections and for all blades together. The thrust force F_t is given by formula 4.12 of KD 35 and this formula is copied as formula 2.

$$F_t = C_t * \tfrac{1}{2} \rho V^2 * \pi R^2 \qquad \text{(N)} \qquad (2)$$

In this formula C_t is the thrust coefficient (-), ρ is the density of air (about 1.2 kg/m^3 at 20 °C at sea level), V is the undisturbed wind speed (m/s) and R is the rotor radius (m). The relation in between the tip speed ratio λ (–), the angular velocity of the rotor Ω (rad/s), the rotor radius R (m) and the undisturbed wind speed V (m/s) is given by formula 1.5 of KD 35.

This formula can be written as:

$$V = \Omega * R / \lambda \qquad (m/s) \qquad (3)$$

(2) + (3) gives:

$$F_t = C_t * ½ \rho * \pi * \Omega^2 * R^4 / \lambda^2 \qquad (N) \qquad (4)$$

Formula 4 gives the rotor thrust for the whole rotor. The rotor thrust on a single blade F_{tbl} for a rotor with a number of blades B is given by:

$$F_{tbl} = C_t * ½ \rho * \pi * \Omega^2 * R^4 / (B * \lambda^2) \qquad (N) \qquad (5)$$

So formula 5 gives the rotor thrust acting on one blade. However, to determine the bending moment caused by this force, one has to know the distribution of the thrust over the blade length. The aerodynamic theory as given in KD 35 is based on the assumption that every small area ΔA of the total swept rotor area πR^2 supplies the same amount power ΔP. Betz has proven that this is the case if the wind speed in the rotor plane is reduced to 2/3 V. This means that there must be a small pressure drop over the rotor plane which is the same for every small area ΔA. Now let’s take a small ring shaped area with a width dr. The area dA of this ring is given by:

$$dA = \pi * r * dr \qquad (m^2) \qquad (6)$$

So the area dA increases proportional to r and this means that the thrust distribution over the blade length is triangular if the pressure difference is the same everywhere. It starts with zero at the centre of the rotor and it is maximal at the blade tip. In practice, the thrust is reduced at the blade tip because of tip losses but this effect is neglected.

The moment caused by a triangular thrust load is the same as for a point load acting at 2/3 of the radius. For the point load one takes the load as found by formula 5. The radius for a cone angle ε is 2/3 R * cos ε (see figure 2). The bending moment caused by the thrust on one blade is called M_t. M_t for a cone angle ε is given by:

$$M_t = F_{tbl} * 2/3 R * \cos \varepsilon \qquad (Nm) \qquad (7)$$

(5) + (7) gives:

$$M_t = C_t * 1/3 \rho * \pi * \Omega^2 * R^5 * \cos \varepsilon / (B * \lambda^2) \qquad (N) \qquad (8)$$

3 Description of the centrifugal force

It is assumed that a blade is made of massive wood provided with a Gö 623 airfoil and that there is blade from the centre of the rotor up to the blade tip. The centre of gravity of the blade length is lying at r = ½ R if the blade isn’t bending backwards. However, it is lying at ½ R * cos ε if the blade has bent backwards with an angle ε. The centre of gravity of the airfoil is lying at about 0.44 c from the airfoil nose. The centrifugal force F_c is given by:

$$F_c = ½ m * R * \cos \varepsilon * \Omega^2 \qquad (N) \qquad (9)$$

If the blade is divided in many identical blade segments with a width dr, the centrifugal force for each blade segment is proportional to r. This means that the distribution of the centrifugal force is also triangular.

The moment caused by a triangular thrust load is the same a for a point load acting at 2/3 of the radius. For the point load one takes the load as found by formula 9. The radius for a cone angle ε is 2/3 R * sin ε. The bending moment caused by the centrifugal force on one blade is called M_c. M_c for a cone angle ε is given by:

$M_c = F_c * 2/3\ R * \sin \varepsilon$ (Nm) (10)

(9) + (10) gives:

$M_c = 1/3\ m * R^2 * \Omega^2 * \cos \varepsilon * \sin \varepsilon$ (Nm) (11)

For the blade mass m it is valid that:

$m = A_{pr} * R * \rho_h$ (kg) (12)

A_{pr} is the cross sectional area of the airfoil (m^2) (pr comes from the Dutch word profile for airfoil), ρ_h is the density of the used wood (kg/m^3) (h comes from the Dutch word hout for wood).

(11) + (12) gives:

$M_c = 1/3 * A_{pr} * \rho_h * R^3 * \Omega^2 * \cos \varepsilon * \sin \varepsilon$ (Nm) (13)

If the blade is connected to the hub by a hinge, it will bend backwards for an angle ε for which $M_t = M_c$. So $M_t = M_c$ gives:

$C_t * 1/3\ \rho * \pi * \Omega^2 * R^5 * \cos \varepsilon / (B * \lambda^2) = A_{pr} * \rho_h * R^3 * \Omega^2 * \cos \varepsilon * \sin \varepsilon$ or

$C_t * \rho * \pi * R^2 = A_{pr} * \rho_h * \sin \varepsilon * B * \lambda^2$ or

$\sin \varepsilon = C_t * \rho * R^2 * \pi / (B * A_{pr} * \rho_h * \lambda^2)$ (14)

This is the same formula as formula 6 of report R 409 D. This formula can also be written as:

$\varepsilon = \arcsin \{C_t * \rho * R^2 * \pi / (B * A_{pr} * \rho_h * \lambda^2)\}$ (°) (15)

The forces and radii are illustrated in figure 1, 2 and 3 of R 409 D. These figures are scanned and given below as fig. 1, 2 and 3. As R 409 D is a Dutch report the inscriptions are given in Dutch. In stead of F_t, τ was used. “Zwaartelijn” means centre line of gravity. “Rotatie as blad” means axis of rotation. Report R 409 D was originally written for a blade with pitch control but for a rotor with fixed blades this isn’t relevant. “Aerodynamische as” means aerodynamic centre (at 0.25 c). “Rotatie as rotor” means axis of rotation of the rotor.

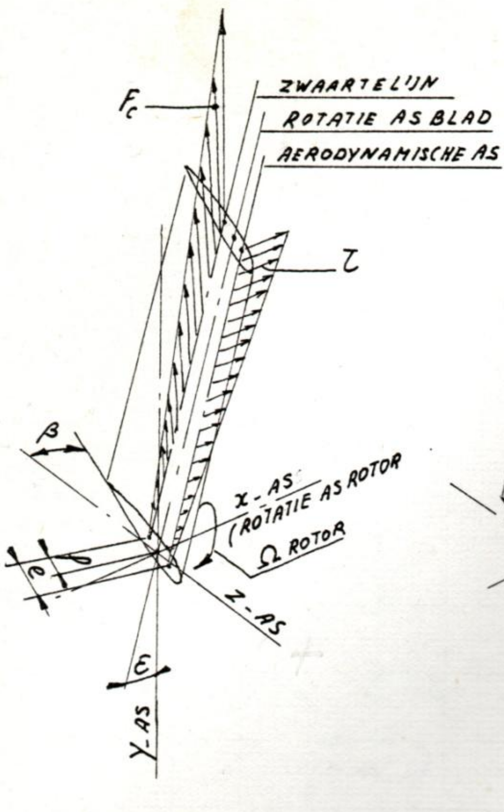

FIG. 1

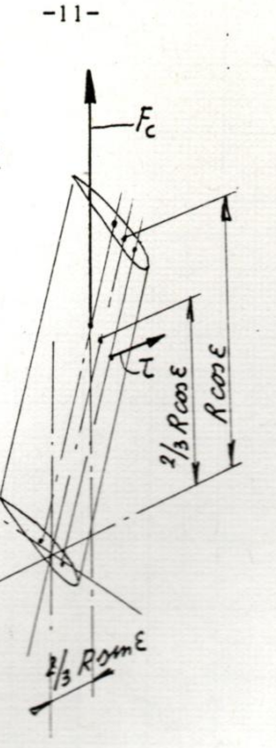

FIG 2

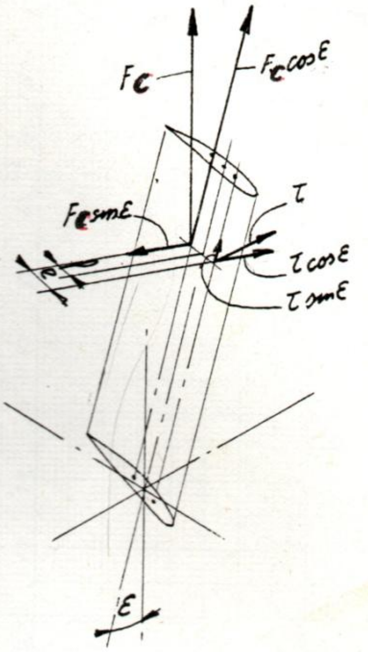

FIG 3

In figure 2 it can be seen that there is a distance in between the thrust force and the centrifugal force because the thrust force exerts at 0.25 c from the nose and because the centrifugal force exerts at about 0.44 c from the nose for a massive wooden blade. Because of this distance, a certain moment is exerted on the blade with has a tendency to reduce the blade angle β. This moment has to be taken into account if the rotor has a pitch control safety system and a blade axis at a cone angle ε. This moment has the correct direction if a negative pitch control or active stall safety system is chosen. For a rotor with fixed blades, this moment isn't relevant.

Apart from the moment caused by the distance e in between F_v sinε and τ cosε, there is also an aerodynamic moment around the aerodynamic centre which has a direction such that the blade angle β is increased. So this moment supports positive pitch control.

4 References

1 Kragten A. Rotor design and matching for horizontal axis wind turbines, January 1999, reviewed February 2017, free public rapport KD 35, engineering office Kragten Design, Populierenlaan 51, 5492 SG Sint-Oedenrode, The Netherlands.

2 Kragten A. Calculations executed for the 3-bladed rotor of the VIRYA-3B3 windmill (λ_d = 6.5, wooden blades), February 2012, reviewed November 2018, free public report KD 484, engineering office Kragten Design, Populierenlaan 51, 5492 SG Sint-Oedenrode, The Netherlands.

3 Kragten A. Bepaling kegelhoek ε en de grootte van de centrifugaalgewichten voor snellopende propellers (in Dutch, translated as: Determination of the cone angle ε and the size of the centrifugal weights for fast running propellers), July 1980, report R 409 D, University of Technology Eindhoven, Faculty of Physics, Laboratory of Fluid

Dynamics and Heat Transfer, P.O. box 513, 5600 MB Eindhoven, The Netherlands, (report no longer available).